Dolphin Semigroups

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2016-04-06

Semigroups

Definition

A **semigroup** is a set S together with a binary operation $* : S \times S \to S$ such that

$$(x * y) * z \quad = \quad x * (y * z)$$

for all $x, y, z \in S$.

- S could be a set of numbers
- S could be a set of transformations
- S could be a set of words
- S could be a set of dolphins

How do you multiply dolphins?

- Stupid question
- I've found three ways (sort of)

First a few words about genetics

- Most Recent Common Ancestor (MRCA)
- Most Recent Common Mitochondrial Ancestor (MRCMA)
- Mitochondrial Eve
- We are our own ancestors

The Pod Semilattice

Definition

Let S be a set of dolphins. S is called a **pod** if it contains a distinguished dolphin m who is a mitochondrial ancestor of all dolphins in S.

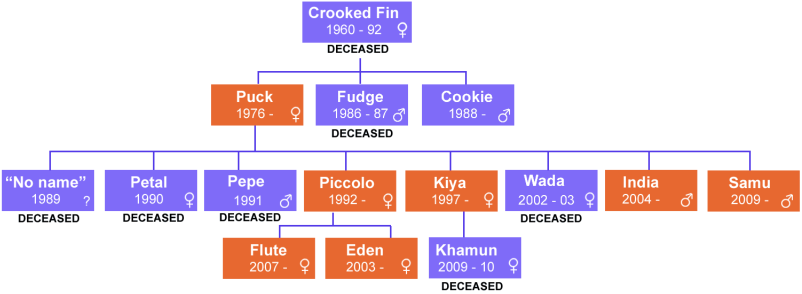
For dolphins $x, y \in S$ let $x \wedge y$ be the MRCMA of x and y.

Theorem

Let S be a non-empty set of dolphins. S is a semigroup under the operation $\wedge$ if and only if S is a pod.

Example: Monkey Mia

A family of bottlenose dolphins in Monkey Mia, Shark Bay, Western Australia:



- Petal $\wedge$ Pepe $=$ Puck
- Eden $\wedge$ Piccolo $=$ Piccolo
- Cookie $\wedge$ Cookie $=$ Cookie
- Khamun $\wedge$ Fudge $=$ Crooked Fin

Semilattices

Two equivalent definitions:

Definition

A **semilattice** is a commutative semigroup whose elements are all idempotents.

Definition

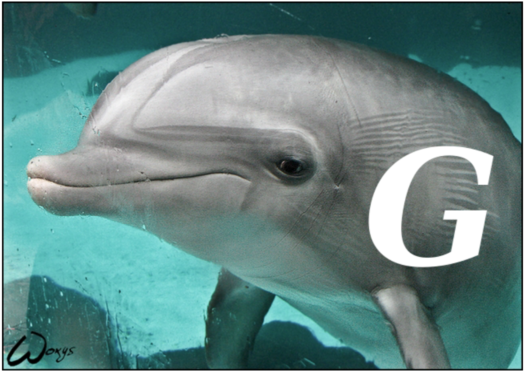
A **semilattice** is a partially-ordered set (poset) such that any pair of elements has a *greatest lower bound* (*meet*).

Theorem

A pod under $\wedge$ *is a semilattice.*

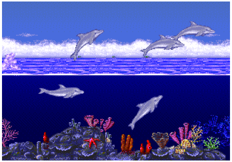
Properties of a Pod Semilattice

- S contains a zero: the mitochondrial ancestor m.
- An ideal of S is a union of dolphins together with all their ancestors.
- S is a group iff it has size 1 (a lonely dolphin)

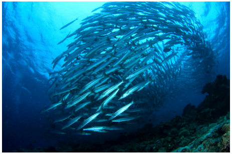


The Gossip Problem

Suppose n dolphins go for a swim one day.



Each dolphin finds a shoal of fish, but keeps it a secret.



Later, any pair of dolphins might bump into each other and start chatting.

At each meeting, both dolphins reveal to each other everything they know.

Gossip Matrices

- We can represent a state of knowledge by a matrix.
- Let K be an $n \times n$ boolean matrix, where $k_{ij} = 1$ iff dolphin j knows about shoal i.
- columns $=$ dolphins, rows $=$ shoals.
- The initial state of knowledge: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- After dolphins 1 and 2 swap information: $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- Or instead, after dolphins 2 and 3 swap information: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Composing Gossip Matrices

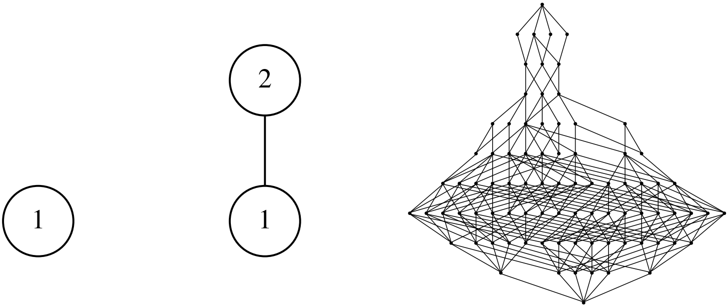
- How about $1 \leftrightarrow 2$ followed by $2 \leftrightarrow 3$?

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

- Composing two exchanges of information is the same as multiplying two boolean matrices.
- The $\binom{n}{2}$ elementary "chats" generate all possible states of information.

The Gossip Monoid

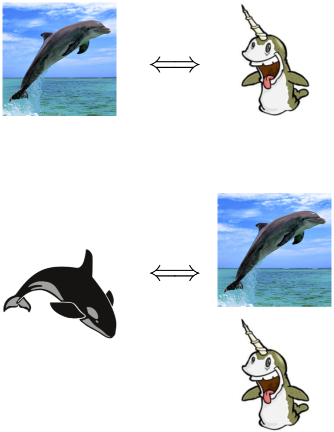
- Hence we have a monoid G_n of boolean matrices, with identity I_n and operation of matrix multiplication.
- Available in **Semigroups** package for GAP.
- It has B_n idempotents (Peter Fenner, University of Manchester, 22nd NBSAN Meeting).
- Its lattice of congruences looks like this:

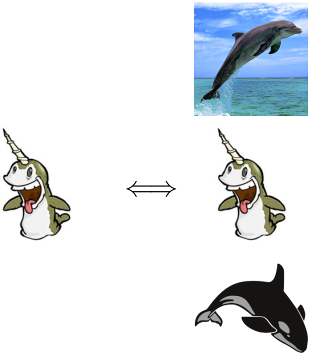


The Mad Marine Biologist

- Suppose a marine biologist has lost her mind.
- She builds a machine to turn certain cetaceans into other cetaceans.
- The **dolphin machine** turns a dolphin into a narwhal.
- The **narwhal machine** turns one narwhal into one dolphin, one narwhal, and one orca.
- The **orca machine** turns one orca into one dolphin, and one narwhal.
- Every machine can operate in reverse.

The Mad Marine Biologist





The Mad Marine Biologist

Let n be the number of machines in our scenario (one for each species).

Definition

A **menagerie** is a tuple in the set

$$S = \mathbb{N}^n \setminus \{\mathbf{0}\}.$$

Example

The mad marine biologist has 4 dolphins, 2 narwhals and no orcas. This corresponds to the menagerie $(4, 2, 0)$.

The Mad Marine Biologist Semigroup

- The set S of menageries forms a semigroup under vector addition.
- $(4, 2, 0) + (0, 1, 1) = (4, 3, 1)$.
- Now consider the relation $\sim$ such that $x \sim y$ iff x can be transformed into y by a sequence of machine transformations.
- $\sim$ is an equivalence relation.
- In fact, $\sim$ is a congruence: $x \sim y, s \sim t \to x + s \sim y + t$.
- Hence we can consider the quotient $W = S/\sim$.
- In our example, $W = \{[(1, 0, 0)], [(2, 0, 0)], [(3, 0, 0)]\} \cong C_3$.

The Mad Marine Biologist Graph

- Any mad marine biologist scenario has a digraph Γ associated to it.
- The vertices correspond to the species, and there are x edges from vertex I to vertex J, where x is the number of cetaceans of species J that are produced by machine I.

Theorem

The mad marine biologist semigroup is a group if and only if its graph Γ fulfils the following:

1. *Every vertex is connected to every cycle,*
2. *Every cycle has an exit.*

Gene Abrams, & Jessica K. Sklar. (2010). The Graph Menagerie: Abstract Algebra and the Mad Veterinarian. *Mathematics Magazine, 83*(3), 168-179. http://doi.org/10.4169/002557010x494814

Thank you for listening