Semilattice Congruences

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Semigroups

Definition

A **semigroup** is a set S together with a binary operation $* : S \times S \to S$ such that

$$(x * y) * z \quad = \quad x * (y * z)$$

for all $x, y, z \in S$.

Semilattices

Two equivalent definitions:

Definition

A **semilattice** is a partially-ordered set (poset) such that any pair of elements has a *greatest lower bound* (*meet*).

Definition

A **semilattice** is a commutative semigroup whose elements are all idempotents.

How are these conditions equivalent?

- Let S be a commutative semigroup of idempotents.
- Define a partial order $\leq$ on S by $a \leq b$ iff $ab = a$.
- Then ab is a lower bound of a and b and it turns out that it is their *greatest lower bound*.
- Conversely, let $(S, \leq)$ be a poset with meets.
- Define a multiplication $\cdot$ on S by $a \cdot b = a \wedge b$.
- Then $(S, \cdot)$ is a commutative semigroup of idempotents.

Congruences

Definition

A **congruence** on a semigroup S is a relation $\rho \subseteq S \times S$ which is

- Reflexive: $(a, a) \in \rho$,
- Symmetric: $(a, b) \in \rho \quad \Rightarrow \quad (b, a) \in \rho$,
- Transitive: $(a, b), (b, c) \in \rho \quad \Rightarrow \quad (a, c) \in \rho$,
- Compatible: $(a, b) \in \rho \quad \Rightarrow \quad (xa, xb), (ax, bx) \in \rho$,

for all $a, b, c, x \in S$.

Definition

Let $(a, b) \in S \times S$. The congruence **generated by** (a, b) is the least congruence ρ on S which contains the pair (a, b). We might write this as $\langle (a, b) \rangle$.

Blocks

- Consider a semilattice S, and a pair $(x, y) \in S \times S$.
- Let ρ be the congruence generated by (x, y).
- ρ has precisely one non-trivial congruence class B_{xy}, which is given by

$$B_{xy} = \{t \in S \mid xy \leq t \leq x \text{ or } xy \leq t \leq y\}.$$

Congruences generated by multiple pairs

- Consider the congruence ρ generated by two pairs: (x, y) and (s, t).
- This is the *join* of the two congruences $\langle(x, y)\rangle$ and $\langle(s, t)\rangle$.
- The non-trivial congruence classes are either B_{xy} and B_{st} (if these are disjoint), or the union $B_{xy} \cup B_{st}$ (if they are not).

Lemma

$B_{xy} \cap B_{st} \neq \varnothing$ if and only if at least one of the following is true:

- $xy \leq xs$ *and* $st \leq xs$,
- $xy \leq xt$ *and* $st \leq xt$,
- $xy \leq ys$ *and* $st \leq ys$,
- $xy \leq yt$ *and* $st \leq yt$.

i.e. if $xy \leq p$ *and* $st \leq p$ *for some* $p \in \{xs, xt, ys, yt\}$.

Block coincidence table

We can use the above to calculate a **block coincidence table**.

- Check each generating pair (x, y) against each other generating pair (s, t) to see if B_{xy} and B_{st} overlap.
- Store the information in a table, so we can quickly check whether two blocks are in the same congruence class.

Congruence class number

To find which congruence class an element a is in, use the following algorithm:

Algorithm

procedure CONGRUENCECLASSNO(ρ, a)
  $R :=$ GENERATINGPAIRS(ρ)
  for $i \in \{1 \ldots |R|\}$ **do**
    Let $(x, y) = R[i]$
    if $xy \leq a$ **and** $(a \leq x$ **or** $a \leq y)$ **then**
      return BLOCKCOINCIDENCETABLE$(\rho)[i]$
    end if
  end for
  return 0 *(singleton)*
end procedure

Two elements a and b are related if and only if $a = b$ or $\textsc{CongruenceClassNo}(a) = \textsc{CongruenceClassNo}(b) \neq 0$.

Thank you