Computing with Semigroup Congruences

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2014-10-31

Semigroups

Definition

A **semigroup** is a set S together with a binary operation $*: S \times S \to S$ such that

$$(x * y) * z \quad = \quad x * (y * z)$$

for all $x, y, z \in S$.

Congruences

Definition

A **congruence** on a semigroup S is a relation $\rho \subseteq S \times S$ such that

(R) $(x, x) \in \rho$,

(S) $(x, y) \in \rho \quad \Rightarrow \quad (y, x) \in \rho$,

(T) $(x, y), (y, z) \in \rho \quad \Rightarrow \quad (x, z) \in \rho$,

(C) $(x, y) \in \rho \quad \Rightarrow \quad (ax, ay), (xa, ya) \in \rho$,

or equivalently,

(C) $(x, y), (s, t) \in \rho \quad \Rightarrow \quad (xs, yt) \in \rho$,

for all $x, y, z, a, s, t \in S$.

(we may write $x \; \rho \; y$ for $(x, y) \in \rho$)

Simple ways to represent congruences

- List of pairs: $\{\{x_1, x_3\}, \{x_1, x_9\}, \{x_{42}, x_{11}\}, \dots\}$
- Partition: $\{\{x_1, x_3, x_9, x_{14}\}, \{x_2\}, \{x_4, x_5, x_8\}, \dots\}$
- ID list: $(1, 2, 1, 3, 3, 4, 5, 3, 1, \dots)$

Generating pairs

Let $\mathbf{R} \subseteq S \times S$ be a set of pairs.

- The relation $\mathbf{R}^e$ given by
$$\left(\mathbf{R} \cup \mathbf{R}^{-1} \cup \Delta_S\right)^{\infty}$$
is the smallest equivalence on S containing $\mathbf{R}$.
- The relation $\mathbf{R}^c$ given by
$$\left\{(xay, xby) \;\middle|\; (a, b) \in \mathbf{R},\ \ x, y \in S^1\right\}$$
is the smallest compatible relation on S containing $\mathbf{R}$.
- The relation $\mathbf{R}^\sharp$ given by
$$(\mathbf{R}^c)^e$$
is the smallest congruence on S containing $\mathbf{R}$.

Ideals

Let S be a semigroup.

Definition

A **left (right) ideal** is a non-empty subset $I \subseteq S$ such that

$$si \in I \quad (is \in I)$$

for all $s \in S$ and $i \in I$.

Definition

An **ideal** is a non-empty subset $I \subseteq S$ which is both a left ideal and a right ideal.

0-simple semigroups

Definition

A semigroup element $0 \in S$ is called **zero** if

$$0x = x0 = 0$$

for all $x \in S$.

Definition

A semigroup S without zero is **simple** if it has no proper ideals.

Definition

A semigroup S with zero is **0-simple** if its only ideals are $\{0\}$ and S.

Rees 0-matrix semigroups

Definition

A **Rees 0-matrix semigroup** $\mathcal{M}^0[T; I, \Lambda; P]$ is the set

$$(I \times T \times \Lambda) \cup \{0\}$$

with multiplication given by

$$(i, a, \lambda) \cdot (j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if } p_{\lambda j} \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where

- T is a semigroup,
- I and Λ are index sets,
- P is a $|\Lambda| \times |I|$ matrix with entries $(p_{\lambda i})_{\lambda \in \Lambda, i \in I}$ taken from T^0,
- $0x = x0 = 0$ for all x in the semigroup.

The Rees Theorem

Theorem (Rees)

Every completely 0-simple semigroup is isomorphic to a Rees 0-matrix semigroup

$$\mathcal{M}^0[G; I, \Lambda; P],$$

where G is a group and P is regular. Conversely, every such Rees 0-matrix semigroup is completely 0-simple.

Linked triples

Definition

For a finite 0-simple Rees 0-matrix semigroup $\mathcal{M}^0[G; I, \Lambda; P]$, a **linked triple** is a triple

$$(N, \mathcal{S}, \mathcal{T})$$

consisting of a normal subgroup $N \trianglelefteq G$, an equivalence relation $\mathcal{S}$ on I and an equivalence relation $\mathcal{T}$ on Λ, such that the following are satisfied:

1. $\mathcal{S}$ only relates columns which have zeroes in the same places,
2. $\mathcal{T}$ only relates rows which have zeroes in the same places,
3. For all $i, j \in I$ and $\lambda, \mu \in \Lambda$ such that $p_{\lambda i}, p_{\lambda j}, p_{\mu i}, p_{\mu j} \neq 0$ and either $(i, j) \in \mathcal{S}$ or $(\lambda, \mu) \in \mathcal{T}$, we have that $q_{\lambda\mu ij} \in N$, where

$$q_{\lambda\mu ij} = p_{\lambda i} p_{\mu i}^{-1} p_{\mu j} p_{\lambda j}^{-1}.$$

Linked triples

A finite 0-simple semigroup S has a bijection Γ between its linked triples and its *non-universal* congruences:

$$\Gamma : \rho \mapsto (N, \mathcal{S}, \mathcal{T})$$

Two non-zero elements (i, a, λ) and (j, b, μ) are ρ-related if and only if

1. $(i, j) \in \mathcal{S}$;
2. $(\lambda, \mu) \in \mathcal{T}$;
3. $(p_{\xi i} a p_{\lambda x})(p_{\xi j} b p_{\mu x})^{-1} \in N$ for some $x \in I, \xi \in \Lambda$ such that $p_{\xi i}, p_{\xi j}, p_{\lambda x}, p_{\mu x} \neq 0$.

Inverse semigroups

Definition

Let S be a semigroup and $x \in S$. An element $y \in S$ is the **inverse** of x if and only if

$$xyx = x \text{ and } yxy = y.$$

Definition

An **inverse semigroup** is a semigroup S such that every element $x \in S$ has a unique inverse $x^{-1} \in S$.

Idempotents

Definition

An **idempotent** is a element $e \in S$ such that

$$ee = e.$$

Theorem

In an inverse semigroup S, the set of idempotents E is a commutative subsemigroup of S.

Trace and kernel

For a congruence ρ on an inverse semigroup S:

Definition

The **trace** of ρ is the restriction of ρ to the idempotents of S.

$$\operatorname{tr} \rho = \rho \cap (E \times E).$$

Definition

The **kernel** of ρ is the union of all the ρ-classes of S which contain idempotents.

$$\operatorname{Ker} \rho = \bigcup_{e \in E} \rho_e.$$

Trace and kernel

Theorem

$(x, y) \in \rho$ *if and only if*

- $(x^{-1}x, y^{-1}y) \in \operatorname{tr} \rho$,
- $xy^{-1} \in \operatorname{Ker} \rho$.

"Congruence pairs"

Definition

A **congruence pair** is a pair (N, τ) consisting of a normal subsemigroup N of S and a normal congruence τ on E, such that

1. If $ae \in N$ and $(e, a^{-1}a) \in \tau$, then $a \in N$
2. If $a \in N$, then $(aa^{-1}, a^{-1}a) \in \tau$

for all elements $a \in S$ and $e \in E$.