Inverse Semigroups

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Semigroups

Definition

A ~~**semigroup**~~ ~~**monoid**~~ **group**
is a set S together with a binary operation $* : S \times S \to S$ which

- is **associative**: $(xy)z = x(yz)$
- has an **identity** e: $ex = xe = x$
- has **inverses** x^{-1}: $xx^{-1} = x^{-1}x = e$

What is an inverse?

In the language of groups:

Definition

An **inverse** of an element x in a group G is some $y \in G$ such that

$$xy = yx = e,$$

where e is the identity.

Clearly in a semigroup, this doesn't apply.
In the language of semigroups:

Definition

An **inverse** of an element x in a semigroup S is some $y \in S$ such that

$$xyx = x \quad \text{and} \quad yxy = y.$$

Note: group inverses are semigroup inverses.

Regularity

Definition

An **inverse** of an element x in a semigroup S is some $y \in S$ such that

$$xyx = x \quad \text{and} \quad yxy = y.$$

Definition

An element $x \in S$ is **regular** if it has an inverse.

Definition

A semigroup S is **regular** if every element has an inverse.

Note: groups are regular.

Inverses aren't unique

- In a group, x has precisely one inverse, x^{-1}.
- In a semigroup, x could have several inverses.
- We just need $y, z \in S$ such that

$$xyx = x, \quad yxy = y, \quad xzx = x, \quad zxz = z.$$

- For example, in the full transformation semigroup T_3, let

$$x = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}.$$

$$xyx = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = x,$$
$$yxy = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix} = y.$$

$$xzx = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = x,$$
$$zxz = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix} = z.$$

So x has at least two inverses, y and z.

Inverse Semigroups

Definition

An **inverse semigroup** is a semigroup in which each element has precisely one inverse.

Example

Groups are inverse semigroups.

Proof.

An element x of a group G has at least one inverse: its *group inverse* x^{-1}. Let y and z be inverses for x. Now, $xyx = x$ and $xzx = x$, so

$$xyx = xzx.$$

This cancels to $xy = xz$ and then to $y = z$. Hence x has precisely one inverse. □

Partial Permutations

Definition

A **permutation** on a set X is a bijection from X to X.

Definition

A **partial permutation** on a set X is a bijection from *a subset of* X to *a subset of* X.

Example

The partial permutation $x = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & - & 2 & - & 5 \end{smallmatrix}\right)$ is a bijection from $\{1, 3, 5\}$ to $\{2, 3, 5\}$.
We can multiply: $x^2 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & - & 2 & - & 5 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & - & 2 & - & 5 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & - & - & - & 5 \end{smallmatrix}\right)$.
We can find a unique inverse: $x^{-1} = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ - & 3 & 1 & - & 5 \end{smallmatrix}\right)$.
This is all independent of what semigroup x is in.

"Cayley's Theorem for Inverse Semigroups"

Theorem (Cayley, for groups)

Every group G is isomorphic to a permutation group.

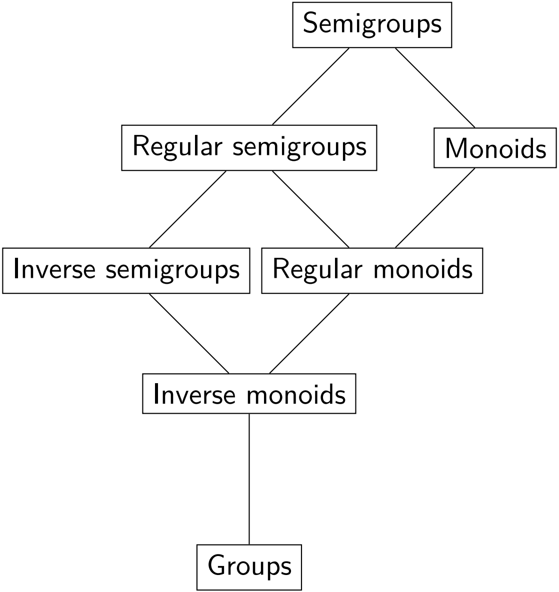
Theorem (Cayley, for semigroups)

Every semigroup S is isomorphic to a transformation semigroup.

Theorem (Wagner-Preston)

Every inverse semigroup I is isomorphic to an inverse semigroup of partial permutations.

Thank you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Groups