Diagram Semigroups

Much more fun than transformations!

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University of St Andrews

2017-06-07

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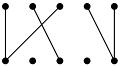
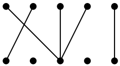
$\alpha =$ , $\beta =$ ,

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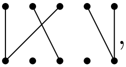
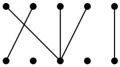
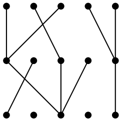
$\alpha =$ , $\beta =$ , $\alpha\beta =$ 

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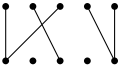
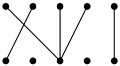
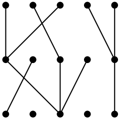
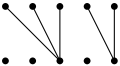
$\alpha =$ , $\beta =$ , $\alpha\beta =$  $=$ 

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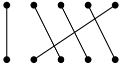
$\alpha =$  ,

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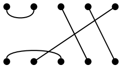
$\alpha =$  ,

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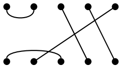
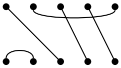
$\alpha =$  , $\beta =$ 

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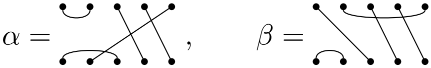

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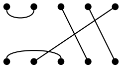
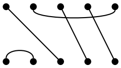
$\alpha =$  , $\beta =$ 

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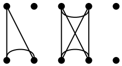
$\gamma =$  ,

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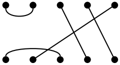
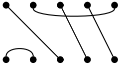
$\alpha =$  , $\beta =$ 

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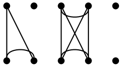
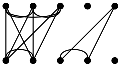
$\gamma =$  , $\delta =$ 

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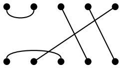
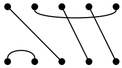
$\alpha =$  , $\beta =$ 

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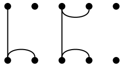
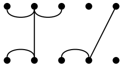
$\gamma =$  , $\delta =$ 

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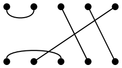
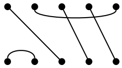
$\alpha =$  , $\beta =$ 

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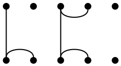
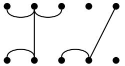
$\gamma =$  , $\delta =$ 

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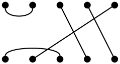
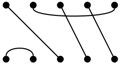
$\alpha =$  , $\quad \beta =$ 

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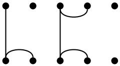
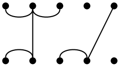
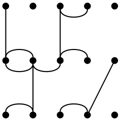
$\gamma =$  , $\quad \delta =$ 

$\gamma\delta =$ 

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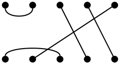
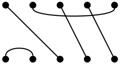
$\alpha =$  , $\beta =$ 

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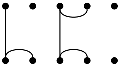
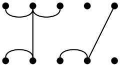
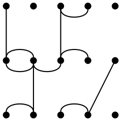
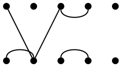
$\gamma =$  , $\delta =$ 

$\gamma\delta =$  $=$ 

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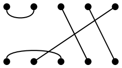
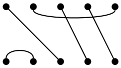
$\alpha =$  , $\beta =$ 

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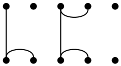
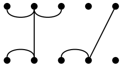
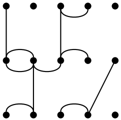
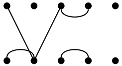
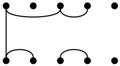
$\gamma =$  , $\delta =$ 

$\gamma\delta =$  $=$  $=$ 

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- $|\mathcal{P}_{10}| \approx 5.2 \times 10^{13}$.

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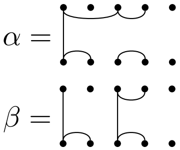

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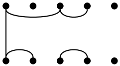
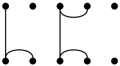
$\alpha =$ 

$\beta =$ 

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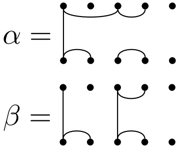

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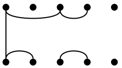
$\alpha =$ 

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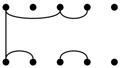
$\alpha =$ 

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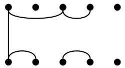
$\alpha =$ 

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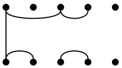
$\alpha =$ 

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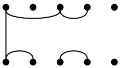
$\alpha =$ 

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[2]

Howie, J.M., *Fundamentals of Semigroup Theory*, Oxford Science Publications, 1995, 1.1, 1.5 & 1.8, 7-35.

James East, Attila Egri-Nagy, Andrew R. Francis, James D. Mitchell, *Finite Diagram Semigroups: Extending the Computational Horizon*, https://arxiv.org/abs/1502.07150