Computing with Semigroup Congruences

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Semigroups

Definition

A **semigroup** is a set S together with a binary operation $* : S \times S \to S$ such that

$$(x * y) * z \quad = \quad x * (y * z)$$

for all $x, y, z \in S$.

Congruences

Definition

A **congruence** on a semigroup S is an equivalence relation on S (a partition of S) such that

- $(x, y) \in \rho \quad \Rightarrow \quad (a * x, a * y), (x * a, y * a) \in \rho,$

or equivalently,

- $(x, y), (s, t) \in \rho \quad \Rightarrow \quad (x * s, y * t) \in \rho,$

for all $x, y, a, s, t \in S$.

(we may write $x \ \rho \ y$ for $(x, y) \in \rho$)

Example

Let $S = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ under addition mod 6.
What congruences do we have?

- Universal congruence: $\{\{0, 1, 2, 3, 4, 5\}\}$
- Trivial congruence: $\{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$
- Three classes: $\{\{0, 3\}, \{1, 4\}, \{2, 5\}\}$
- Two classes: $\{\{0, 2, 4\}, \{1, 3, 5\}\}$
- Not a congruence: $\{\{0, 1\}, \{2, 3, 5\}, \{4\}\}$
  $(2, 3) \in \rho$ but $(1 + 2, 1 + 3) = (3, 4) \notin \rho$.

Representing congruences on a computer

- List of pairs: $\{(x_1, x_3), (x_1, x_9), (x_{42}, x_{11}), \dots\}$
- Partition: $\{\{x_1, x_3, x_9, x_{14}\}, \{x_2\}, \{x_4, x_5, x_8\}, \dots\}$
- ID list: $(1, 2, 1, 3, 3, 4, 5, 3, 1, \dots)$

Abstractions from congruences

- Groups: **normal subgroups**
- Rings: **two-sided ideals**
- Simple & 0-simple semigroups: **linked triples**
- Inverse semigroups: **kernel and trace**

Rees congruences considered separately: store a **semigroup ideal**

Semigroups package

- Package for GAP computational algebra system
- Developed in St Andrews
- Several ways to deal with congruences
- Basic way is using **generating pairs**
- *Abstractions* from previous slide used when appropriate

Thank you